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Noether's theorem and accidental degeneracy*

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Abstract. It is shown that the determination of the symmetry Lie algebra of a quantum system with accidental degeneracy can be obtained, in principle, by using Noether's theorem. The procedure is illustrated by considering a quantum system which represents a generalization of the degeneracies present in an anisotropic two-dimensional harmonic oscillator. Thus, besides the u(2) algebra which gives the fundamental degeneracy of a two-dimensional oscillator, the studied system can have an infinite set of states with the same energy characterized by an u(1, 1) Lie algebra.

1. Introduction

The two classical examples which show accidental degeneracy are the harmonic oscillator and the Coulomb potentials [1, 2]. In both cases, the apparent symmetry of the problem is spherical SO(3) but there are additional non-geometric degeneracies which are observed to occur, they are called accidental. In the present paper we are interested in the study of the accidental degeneracy of the Hamiltonian

$$H = \frac{1}{2}\sum_i (p_i^2 + x_i^2) + \lambda M \tag{1.1}$$

which is a two-dimensional harmonic oscillator plus the projection of the angular momentum in the z direction, M. We use atomic units, in which $\hbar = m = e = 1$ and λ is a constant parameter. The quantum system (1.1) for $\lambda = 1$ describes the motion of an electron in a constant magnetic field [3, 4], and its symmetry Lie algebra has been found recently by Moshinsky *et al* [4].

In this analysis we are going to establish a procedure using Noether's theorem [5] to get the symmetry algebra of the Hamiltonian systems (1.1). This is done for rational values of the parameter λ which include the cases discussed by Moshinsky *et al.* Also we want to stress that the Hamiltonian (1.1) represents a generalization of the degeneracies present in the anisotropic two-dimensional harmonic oscillator [6, 7].

For the purposes of this paper it is convenient to introduce the creation and annihilation operators

$$\eta_i = \frac{1}{\sqrt{2}}(x_i - \mathrm{i}p_i) \qquad \xi_i = \frac{1}{\sqrt{2}}(x_i + \mathrm{i}p_i) \qquad i = 1, 2 \tag{1.2}$$

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as well as the corresponding operators in component form denoted by the indices $\pm$ which are defined as

$$\eta_\pm = \frac{1}{\sqrt{2}}(\eta_1 \pm \mathrm{i}\eta_2) \qquad \xi_\pm = \frac{1}{\sqrt{2}}(\xi_1 \mp \mathrm{i}\xi_2) \tag{1.3}$$

with the properties

$$[\xi_a, \eta_b] = \delta_{ab} \tag{1.4a}$$

$$[\xi_a, \xi_b] = [\eta_a, \eta_b] = 0 \qquad a, b = +, -. \tag{1.4b}$$

Besides the Hermitean conjugate of $(\eta_\pm)^+ = \xi_\pm$. It is straightforward to find the expression of the Hamiltonian (1.1) in terms of these operators

$$H = (1+\lambda)N_+ + (1-\lambda)N_- \tag{1.5}$$

where a constant term was neglected and N_a; $a = \pm$, denotes the number of quanta in direction a. The eigenstates of (1.5) are well known and can be denoted by the kets

$$|n_+, n_-\rangle = \frac{1}{\sqrt{n_+! n_-!}}(\eta_+)^{n_+}(\eta_-)^{n_-}|0\rangle \tag{1.6}$$

with $\xi_\pm|0\rangle = 0$. These states in terms of the polar coordinates representation (ρ, ϕ) are given by

$$\langle\rho, \phi|n_+, n_-\rangle = (-1)^{\frac{1}{2}(\nu-|m|)}\left[\frac{2\left(\frac{1}{2}(\nu-|m|)\right)!}{\pi\left(\frac{1}{2}(\nu+|m|)\right)!}\right]^{\frac{1}{2}} \rho^{|m|}\mathrm{e}^{-\frac{1}{2}\rho^2} L^{|m|}_{\frac{1}{2}(\nu-|m|)}(\rho^2)\mathrm{e}^{\mathrm{i}m\phi} \tag{1.7}$$

where $\nu = n_+ + n_-$; $m = n_+ - n_-$ and $L^{|m|}_n(\rho^2)$ is a Laguerre polynomial.

The eigenvalues of (1.5) can be denoted by

$$E_{\nu m} = (1+\lambda)n_+ + (1-\lambda)n_- = \nu + \lambda m \tag{1.8}$$

with $|m| = \nu, \nu-2 \ldots 1$ or 0. From this expression for the energies, it is immediately possible to see that (1.8) will present degeneracy for rational values of λ, which in general can be denoted as follows:

$$\lambda = -\frac{\Delta\nu}{\Delta m} = -\frac{\nu_f - \nu_i}{m_f - m_i}. \tag{1.9}$$

Thus the accidental degeneracy associated with the Hamiltonian (1.5) can be classified according to the strength of the parameter λ in three groups

$$\lambda = \pm 1 \tag{1.10a}$$

$$\lambda > 1 \qquad \lambda < -1 \tag{1.10b}$$

$$-1 < \lambda < 1. \tag{1.10c}$$

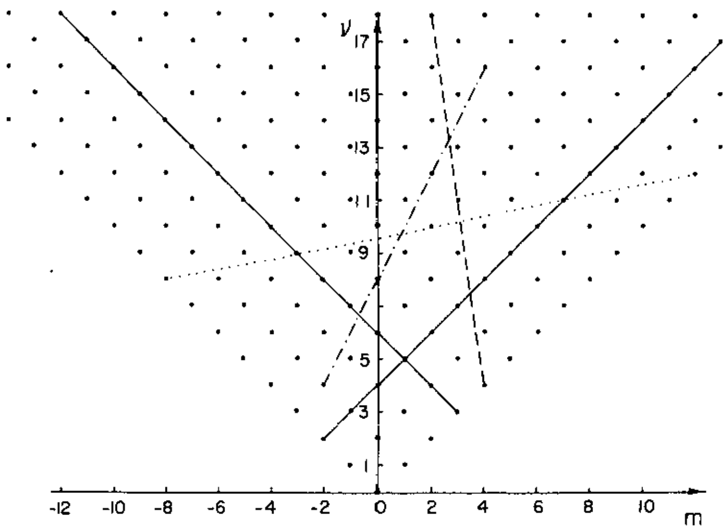

Figure 1. Examples of the degeneracy present in Hamiltonian (1.5) for the cases: (i) $\lambda = 1$, $\lambda = -1$, full line; (ii) $\lambda < -1$, dot-dashed line; (iii) $\lambda > 1$, dashed line; and (iv) $-1 < \lambda < 1$, dotted line.

For the cases (1.10*a*) and (1.10*b*), we can observe from figure 1 an infinite number of levels with the same energy, while for the case (1.10*c*), which includes the value $\lambda = 0$, there is a finite number of levels with the same energy.

Now we turn our attention to the case (1.10*c*) with the anisotropic two-dimensional harmonic oscillator. Comparing equation (1.5) with equation (2.5) of [6], we see, except for constant terms, that they are equivalent systems if we make the identification

$$\kappa_+ = \frac{r}{|\Delta m - \Delta\nu|} \tag{1.11a}$$

$$\kappa_- = \frac{r}{|\Delta m + \Delta\nu|} \tag{1.11b}$$

with

$$r = \mathrm{LCM}(|\Delta m + \Delta\nu|,\ |\Delta m - \Delta\nu|). \tag{1.11c}$$

$\mathrm{LCM}(n_1, n_2)$ defines the lowest common multiple of the positive integers n_1 and n_2. By using equations (1.11) it is easy to show that κ_+ and κ_- are relative prime integers. Of course, the parameter λ must be restricted to rational values in the interval $-1 < \lambda < 1$, which means $|\Delta\nu| < |\Delta m|$. In this sense, we say the Hamiltonian (1.5) is a generalization of the two-dimensional anisotropic oscillator, because it has additional accidental degeneracy for the cases when λ takes the values indicated in (1.10*a*) and (1.10*b*). These cases are equivalent to considering negative or zero frequencies for the two oscillators.

In [7] it was shown that for each λ there are $\kappa_+\kappa_-$ copies of the fundamental degeneracy pattern associated with the two-dimensional isotropic harmonic oscillator. In the section 2, we describe Noether's theorem and consider as an example a one-dimensional Hamiltonian of the form $T + V(q)$. In section 3 by using the Noether theorem we find the classical symmetry Lie algebra of the generalized two-dimensional anisotropic harmonic oscillator. In the section 4, we discuss, for all the cases of λ, the corresponding symmetry Lie algebras which are responsible of the accidental degeneracy of the Hamiltonian (1.5). Finally some conclusions and remarks are made.

2. Description of the Noether's theorem

We are going to use Noether's theorem in its active version [8]. Given a Lagrangian, which is, in general, a function of the coordinates, velocities and time

$$L = L(q^i, \dot{q}^i, t) \tag{2.1}$$

then an arbitrary infinitesimal transformation of the coordinates q^i

$$q^i \to q^i + \delta q^i \tag{2.2}$$

is a symmetry transformation if the variation induced on the Lagrangian can be written as a total time derivative of a function Ω

$$\delta L = \frac{\partial L}{\partial \dot{q}^i}(\delta q^i)^{\cdot} + \frac{\partial L}{\partial q^i}(\delta q^i) \equiv \frac{\mathrm{d}\Omega}{\mathrm{d}t}. \tag{2.3}$$

Noether's theorem states that to the symmetry transformation (2.2) there corresponds a constant of the motion, of the Noether charge, given by

$$K = \sum_i \left(\frac{\partial L}{\partial \dot{q}^i}\right) \delta q^i - \Omega. \tag{2.4}$$

In this work we are interested in considering only time independent variations δq^i, since these give rise to energy preserving symmetries, i.e. conserved quantities whose Poisson bracket with the Hamiltonian vanishes. It is easy to show that these constants of the motion are closed among themselves and form a Lie algebra under the Poisson bracket operation [9]. In general the algebra is infinite-dimensional but we are interested only in the minimal subset of constants of the motion which are closed and the associated quantum operators must connect all the states with the same energy. This minimal set of constants of motion define the symmetry algebra for the system. As an example let us consider a one-dimensional system whose Lagrangian is given by

$$L = \tfrac{1}{2}\dot{q}^2 - V(q). \tag{2.5}$$

It is well known that a variation of the form $\delta q = \epsilon \dot{q}$ gives a Noether charge proportional to the Hamiltonian of the system. If we propose a general transformation

$$\delta q = F(q, \dot{q}) \tag{2.6}$$

where F is an arbitrary function of the coordinate q and velocity $\dot{q}$, the corresponding variation induced in the Lagrangian is

$$\delta L = \dot{q}\left(\frac{\partial F}{\partial \dot{q}}\ddot{q} + \frac{\partial F}{\partial q}\dot{q}\right) - \frac{\partial V}{\partial q}F. \tag{2.7}$$

If we want (2.6) to be a symmetry transformation, we must find a function $\Omega = \Omega(q, \dot{q})$ such that

$$\delta L = \dot{\Omega} = \frac{\partial \Omega}{\partial \dot{q}} \ddot{q} + \frac{\partial \Omega}{\partial q} \dot{q}. \tag{2.8}$$

Comparing expressions (2.7) and (2.8) one finds that the following system of partial differential equations must be satisfied

$$\frac{\partial \Omega}{\partial \dot{q}} = \dot{q} \frac{\partial F}{\partial \dot{q}} \tag{2.9a}$$

$$\frac{\partial \Omega}{\partial q} = \dot{q} \frac{\partial F}{\partial q} - \frac{1}{\dot{q}} \frac{\partial V}{\partial q} F. \tag{2.9b}$$

The integrability condition for this system, namely equality in crossed partial derivatives, yields a linear partial differential equation of first-order for F

$$\dot{q}^2 \frac{\partial F}{\partial q} - \dot{q} \frac{\partial V}{\partial q} \frac{\partial F}{\partial \dot{q}} + \frac{\partial V}{\partial q} F = 0. \tag{2.10}$$

This equation can be solved through the characteristics method [10]. Solving the system

$$\frac{\mathrm{d}q}{\dot{q}^2} = \frac{\mathrm{d}\dot{q}}{-\dot{q}\left(\frac{\partial V}{\partial q}\right)} = -\frac{\mathrm{d}F}{F\left(\frac{\partial V}{\partial q}\right)} \tag{2.11}$$

we get

$$F = \dot{q} G\left(\tfrac{1}{2}\dot{q}^2 + V(q)\right) \tag{2.12}$$

where G in an arbitrary function of the Hamiltonian H. Integrating (2.9*a*) we have

$$\Omega = \dot{q} F(q, \dot{q}) - \int^{\dot{q}} \mathrm{d}\dot{q}'\, F(q, \dot{q}') + \phi(q) \tag{2.13}$$

where $\phi(q)$ is an arbitrary function. Substituting (2.13) into equation (2.9*b*) and using (2.12), it can be shown that $\phi(q)$ is a constant that can be neglected. The corresponding constant of motion is

$$K = \dot{q} F(q, \dot{q}) - \Omega = \int^{H} \mathrm{d}H'\, G(H') \tag{2.14}$$

that is, an arbitrary function of the Hamiltonian. This result implies that for a classical system in one-dimension the symmetry algebra has only one generator, the Hamiltonian.

3. Classical symmetry algebra for the Hamiltonian

In this section we apply Noether's theorem to the system described by Hamiltonian (1.5). Therefore we must obtain the corresponding Lagrangian. One of the Hamilton equations tells us

$$\dot{x}_a = \frac{\partial H}{\partial p_a} = \lambda_a p_a \tag{3.1}$$

where to simplify the notation we associate indices 1 and 2 with the labels + and −, and we defined $\lambda_1 = 1 + \lambda$ and $\lambda_2 = 1 - \lambda$. From here on we adopt a modified summation convention: repeated indices are summed except when one of them appears with λ; for example, in equation (3.1) there is no sum over index a. The Lagrangian is defined in terms of the Hamiltonian as

$$L = \dot{x}_a p_a - H = \frac{1}{2\lambda_a}\left(\dot{x}_a^2 - \lambda_a^2 x_a^2\right). \tag{3.2}$$

Let us propose a symmetry transformation in terms of an arbitrary function of coordinates and velocities

$$\delta x_a = F_a(x_b, \dot{x}_b). \tag{3.3}$$

The corresponding variation induced in the Lagrangian (3.2) is

$$\delta L = (\delta x_a)^{\cdot}\frac{\partial L}{\partial \dot{x}_a} + (\delta x_a)\frac{\partial L}{\partial x_a} = \left(\ddot{x}_b \frac{\partial F_a}{\partial \dot{x}_b} + \dot{x}_b \frac{\partial F_a}{\partial x_b}\right)\frac{1}{\lambda_a}\dot{x}_a - F_a \lambda_a x_a . \tag{3.4}$$

Because (3.3) is a symmetry transformation of the system, the last expression must be a total time derivative

$$\frac{d\Omega}{dt} = \ddot{x}_b \frac{\partial \Omega}{\partial \dot{x}_b} + \dot{x}_b \frac{\partial \Omega}{\partial x_b} \tag{3.5}$$

implying that the following system of equations must be satisfied

$$\frac{\partial \Omega}{\partial \dot{x}_a} = \frac{1}{\lambda_b}\dot{x}_b \frac{\partial F_b}{\partial \dot{x}_a} \tag{3.6}$$

$$\dot{x}_a \frac{\partial \Omega}{\partial x_a} = \frac{1}{\lambda_b}\dot{x}_b \dot{x}_a \frac{\partial F_b}{\partial x_a} - F_a \lambda_a x_a . \tag{3.7}$$

In order to establish the consistency conditions for this system, we must solve for $\partial\Omega/\partial x_a$. For this we derive (3.6) with respect to x_c and (3.7) with respect to $\dot{x}_c$, and compare the results to obtain

$$\frac{\partial \Omega}{\partial x_b} = \frac{1}{\lambda_b}\frac{\partial F_b}{\partial x_c}\dot{x}_c + \frac{1}{\lambda_c}\frac{\partial F_c}{\partial x_b}\dot{x}_c - \lambda_c x_c \frac{\partial F_c}{\partial \dot{x}_b}. \tag{3.8}$$

Now the integrability conditions can be established by asking for equality between the five crossed partial derivatives of Ω. This gives rise to the following system of second-order partial differential equations

$$\frac{\partial F_a}{\partial p_b} = \frac{\partial F_b}{\partial p_a} \tag{3.9a}$$

$$\frac{1}{2}\mathcal{O}\left(\frac{\partial F_a}{\partial p_b} + \frac{\partial F_b}{\partial p_a}\right) - \left(\lambda_a \frac{\partial F_b}{\partial x_a} + \lambda_b \frac{\partial F_a}{\partial x_b}\right) = 0 \tag{3.9b}$$

$$\mathcal{O}\left(\lambda_a \frac{\partial F_b}{\partial x_a} - \lambda_b \frac{\partial F_a}{\partial x_b}\right) - \left(\lambda_a^2 \frac{\partial F_a}{\partial p_b} - \lambda_b^2 \frac{\partial F_b}{\partial p_a}\right) = 0 \tag{3.9c}$$

where the change from velocities to momenta $\dot{x}_a = \lambda_a p_a$ was made, and we have defined the differential operator as

$$\mathcal{O} = \lambda_c \left(x_c \frac{\partial}{\partial p_c} - p_c \frac{\partial}{\partial x_c}\right).$$

From equation (3.9*a*) it is immediately seen that $F_k = \partial G/\partial p_k$. This means that *the function G is the generator of the symmetry transformation.* Through the change of variables

$$z_k = \frac{1}{\sqrt{2}}(x_k + \mathrm{i}p_k)$$

and its complex conjugate, z_k^*, it is immediately seen that the operator $\mathcal{O}$ takes the form

$$\mathcal{O} = i\sum_k \lambda_k (\mathcal{N}_k - \mathcal{N}_k^*) \tag{3.10}$$

with $\mathcal{N}_k = z_k \partial/\partial z_k$. Using the results indicated above, the differential equations (3.9*b*) and (3.9*c*) can be rewritten in the following form

$$\begin{aligned}
&\delta_1 \{\lambda_1(\mathcal{N}_1 - \mathcal{N}_1^* + 1) + \lambda_2(\mathcal{N}_2 - \mathcal{N}_2^* + 1)\} \frac{\partial^2 G}{\partial z_1 \partial z_2} \\
&\quad - \{\lambda_1(\mathcal{N}_1 - \mathcal{N}_1^* + 1) + \lambda_2(\mathcal{N}_2 - \mathcal{N}_2^* - 1)\} \frac{\partial^2 G}{\partial z_1 \partial z_2^*} \\
&\quad - \delta_2 \{\lambda_1(\mathcal{N}_1 - \mathcal{N}_1^* - 1) + \lambda_2(\mathcal{N}_2 - \mathcal{N}_2^* + 1)\} \frac{\partial^2 G}{\partial z_1^* \partial z_2} \\
&\quad + \delta_3 \{\lambda_1(\mathcal{N}_1 - \mathcal{N}_1^* - 1) + \lambda_2(\mathcal{N}_2 - \mathcal{N}_2^* - 1)\} \frac{\partial^2 G}{\partial z_1^* \partial z_2^*} = 0
\end{aligned} \tag{3.11a}$$

$$\begin{aligned}
&\{\lambda_1(\mathcal{N}_1 - \mathcal{N}_1^* + 2) + \lambda_2(\mathcal{N}_2 - \mathcal{N}_2^*)\} \frac{\partial^2 G}{\partial z_k^2} \\
&\quad - 2\{\lambda_1(\mathcal{N}_1 - \mathcal{N}_1^*) + \lambda_2(\mathcal{N}_2 - \mathcal{N}_2^*)\} \frac{\partial^2 G}{\partial z_k \partial z_k^*} \\
&\quad + \{\lambda_1(\mathcal{N}_1 - \mathcal{N}_1^* - 2) + \lambda_2(\mathcal{N}_2 - \mathcal{N}_2^*)\} \frac{\partial^2 G}{\partial z_k^{*2}} = 0 \\
&\quad k = 1, 2
\end{aligned} \tag{3.11b}$$

where the parameters δ_k, $k = 1,2,3$, take two sets of values: $(\delta_1, \delta_2, \delta_3) = (\lambda, -1, -\lambda)$ or $(1,1,1)$. To solve equations (3.11) we propose a solution in terms of powers of the variables z_k and z_k^*

$$G(z_k, z_k^*) = z_1^{n_1} z_2^{n_2} z_1^{*n_3} z_2^{*n_4}$$

where $n_i, i = 1,2,3,4$ are integer numbers. Substitution of this expression into (3.11a) and (3.11b) yields the conditions

$$n_1 = n_3 \qquad n_2 = n_4 \qquad \text{or} \qquad \frac{(n_1 - n_3)}{(n_2 - n_4)} = -\frac{\lambda_2}{\lambda_1} = \frac{\Delta m + \Delta\nu}{-\Delta m + \Delta\nu} \equiv \frac{k_1}{\epsilon k_2} \tag{3.12}$$

where in the last equality the equation (1.9) was used. The integers k_1 and k_2 are relatively prime integers, and the parameter ϵ takes the value 1 or -1. It takes the value 1 when $\Delta m + \Delta\nu$ and $\Delta\nu - \Delta m$ have the same sign, and -1 otherwise. Thus we get, besides the trivial solution, six fundamental solutions, although only three of them are independent. Then the corresponding conserved quantities are given by

$$\begin{aligned} N_1 &= z_1 z_1^* \qquad & K_3 &= z_1^{*k_1} z_2^{-\epsilon k_2} \qquad & K_5 &= z_1^{*k_1} z_2^{*\epsilon k_2} \\ N_2 &= z_2 z_2^* \qquad & K_4 &= z_1^{k_1} z_2^{*-\epsilon k_2} \qquad & K_6 &= z_1^{k_1} z_2^{\epsilon k_2}. \end{aligned} \tag{3.13}$$

From this set we must find a symmetry algebra for the classical system. It is important to realize that to build the algebra once we select a conserved quantity its complex conjugate must be included. Therefore, from (3.13) we must choose three independent constants of motion and their complex conjugates. To do this, it is convenient to find separately, for the cases indicated in equations (1.10), the corresponding expressions for the constants of motion and from them select the independent constants which allows its extension to the quantum case.

For $\lambda = 1$ and $\lambda = -1$ the sets are given by $\{\mathbb{1}, N_1, z_2, z_2^*\}$ and $\{\mathbb{1}, N_2, z_1, z_1^*\}$ respectively. In order to identify the symmetry algebra, we calculate its Poisson brackets, and clearly they correspond to the direct sum of one-dimensional Weyl and unitary algebras, $\mathrm{w}(1) \oplus \mathrm{u}(1)$.

For the cases $\lambda > 1$ and $\lambda < -1$, the constants of motion are identical to each other and from them we choose the set

$$h_1 = \frac{1}{k_1} N_1 - \frac{1}{k_2} N_2 \tag{3.14a}$$

$$m_1 = \frac{1}{k_1 - k_2} (N_1 - N_2) \tag{3.14b}$$

$$K_5 = z_1^{*k_1} z_2^{*k_2} \tag{3.14c}$$

$$K_6 = z_1^{k_1} z_2^{k_2} \tag{3.14d}$$

where $\epsilon = 1$. The constants h_1 and m_1 are proportional to the Hamiltonian and the third component of the angular momentum, respectively. The calculation of the Poisson brackets gives

$$\begin{aligned} \{m_1, K_5\} &= -iK_5 \\ \{m_1, K_6\} &= iK_6 \\ \{K_5, K_6\} &= i\left(k_1^2 N_2 + k_2^2 N_1\right) N_1^{k_1 - 1} N_2^{k_2 - 1} \end{aligned} \tag{3.15}$$

and all other are zero. In order to identify a Lie algebra, the constants of motion K_5 and K_6 are multiplied by functions of N_1 and N_2, whose Poisson bracket vanishes, as follows:

$$\tilde{K}_i = F_i(N_1, N_2) K_i \qquad i = 5, 6. \tag{3.16}$$

The functions F_5 and F_6 are defined in such a way to obtain that the Poisson bracket is

$$\{\tilde{K}_5, \tilde{K}_6\} = iCm_1 \tag{3.17}$$

where C is a constant that can be ± 1. This condition implies that (cf appendix A)

$$F_5 F_6 \doteq \frac{C}{2(k_1 - k_2)^2} (N_1 - N_2)^2 N_1^{-k_1} N_2^{-k_2}. \tag{3.18}$$

Thus one concludes that the set of constants of motion $\{h_1, m_1, \tilde{K}_5, \tilde{K}_6\}$ constitute the classical symmetry Lie algebra for this case which, depending on the value of C, can be identified with a u(2) or u(1,1) algebra.

For $-1 < \lambda < 1$ we select the following independent constants of motion:

$$h_2 = \frac{1}{k_1} N_1 + \frac{1}{k_2} N_2 \tag{3.19a}$$

$$m_2 = \frac{1}{k_1 + k_2} (N_1 - N_2) \tag{3.19b}$$

$$K_3 = z_1^{*k_1} z_2^{k_2} \tag{3.19c}$$

$$K_4 = z_1^{k_1} z_2^{*k_2} \tag{3.19d}$$

where the value $\epsilon = -1$ was used and a combination of N_1 and N_2 proportional to the Hamiltonian, h_2, and angular momentum, m_2, were selected. The Poisson brackets between the constants (3.19) are

$$\begin{aligned} \{m_2, K_3\} &= -iK_3 \\ \{m_2, K_4\} &= iK_4 \\ \{K_3, K_4\} &= -i\left(k_2^2 N_1 - k_1^2 N_2\right) N_1^{k_1 - 1} N_2^{k_2 - 1} \end{aligned} \tag{3.20}$$

and all the others are zero. Proceeding in the same way as in the previous case we can normalize K_3 and K_4 through functions of N_1 and N_2, namely

$$\tilde{K}_i = F_i(N_1, N_2) K_i \qquad i = 3, 4. \tag{3.21}$$

Functions F_3 and F_4 are defined to give the Poisson bracket

$$\{\tilde{K}_3, \tilde{K}_4\} = iCm_2 \tag{3.22}$$

where C is ± 1. This condition implies that

$$F_3 F_4 = \frac{C}{2(k_1 + k_2)^2} (N_1 - N_2)^2 N_1^{-k_1} N_2^{-k_2}. \tag{3.23}$$

Therefore the set of constants of motion $\{h_2, m_2, \tilde{K}_3, \tilde{K}_4\}$ generates the classical symmetry Lie algebras u(2) or u(1,1), depending if the value of C is $+1$ or -1, respectively.

4. Quantum symmetry algebra for the Hamiltonian

To quantize the system we replace the classical variables x and p by the corresponding quantum operators in definitions (3.13), and Poisson brackets by commutators, i.e. $\{\} \to \frac{1}{i}[\,]$. Then the classical variables z_k and z_k^* are replaced by the operators

$$\hat{z}_k = \frac{1}{\sqrt{2}}(\hat{x}_k + \mathrm{i}\hat{p}_k) \qquad \hat{z}_k^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_k - \mathrm{i}\hat{p}_k) \tag{4.1a}$$

which satisfy the commutation relations of creation and annihilation operators

$$[\hat{z}_{k'}, \hat{z}_k^\dagger] = \delta_{kk'}. \tag{4.1b}$$

We choose as a complete base for the physical space the simultaneous eigenstates of $\{N_1, N_2\}$, which we label as $|n_1, n_2\rangle$, because they form a complete set of commuting operators. This lets us see that not all operators in (3.13) make sense at all times. According to section 3 we are led to consider three cases:

(i) For $\lambda = \pm 1$, we have two sets of operators, $\{\mathbb{1}, \hat{N}_1, \hat{z}_2, \hat{z}_2^\dagger\}$ and $\{\mathbb{1}, \hat{N}_2, \hat{z}_1, \hat{z}_1^\dagger\}$, whose commutation relations correspond to the direct sum $\mathrm{w}(1) \oplus \mathrm{u}(1)$.

(ii) When $\lambda > 1$ and $\lambda < -1$, the set of constants of motion (3.14) is replaced by its quantum version, and their commutators are given by

$$[\hat{m}_1, \hat{K}_5] = \hat{K}_5 \qquad [\hat{m}_1, \hat{K}_6] = -\hat{K}_6 \tag{4.2a}$$

$$[\hat{K}_5, \hat{K}_6] = \frac{\hat{N}_1!}{(\hat{N}_1 - k_1)!}\frac{\hat{N}_2!}{(\hat{N}_2 - k_2)!} - \frac{(\hat{N}_1 + k_1)!}{\hat{N}_1!}\frac{(\hat{N}_2 + k_2)!}{\hat{N}_2!} \tag{4.2b}$$

and all the others commutators are zero. The algebra is closed, but to identify a symmetry Lie algebra we must redefine the operators $\hat{K}_5$ and $\hat{K}_6$. Using the properties

$$\begin{aligned} F(\hat{N}_i)\hat{z}_i &= \hat{z}_i F(\hat{N}_i - 1) \\ F(\hat{N}_i)\hat{z}_i^\dagger &= \hat{z}_i^\dagger F(\hat{N}_i + 1) \end{aligned} \tag{4.3}$$

we obtain that the new operators

$$\bar{z}_i^\dagger = \left(F(\hat{N}_i)\frac{(\hat{N}_i - k_i)!}{(\hat{N}_i)!}\right)^{\frac{1}{2}} \hat{z}_i^{\dagger\, k_i} \qquad i = 1, 2 \tag{4.4a}$$

whose Hermitian conjugates are

$$\bar{z}_i = \hat{z}_i^{k_i}\left(F(\hat{N}_i)\frac{(\hat{N}_i - k_i)!}{(\hat{N}_i)!}\right)^{\frac{1}{2}} \qquad i = 1, 2 \tag{4.4b}$$

which satisfy the commutation relations

$$[\bar{z}_i, \bar{z}_j^\dagger] = \{F(\hat{N}_i + k_i) - F(\hat{N}_i)\}\delta_{ij} \qquad i, j = 1, 2. \tag{4.5}$$

Redefining the operators $\tilde{K}_5$ and $\tilde{K}_6$ as

$$\tilde{K}_5 = \tilde{z}_1^\dagger \tilde{z}_2^\dagger \qquad \tilde{K}_6 = \tilde{z}_1 \tilde{z}_2 \tag{4.6}$$

and taking into account that $\tilde{N}_k = \tilde{z}_k^\dagger \tilde{z}_k = \hat{z}_k^\dagger \hat{z}_k = \hat{N}_k$ results in the commutator

$$[\tilde{K}_5, \tilde{K}_6] = -(F(\hat{N}_2 + k_2) - F(\hat{N}_2))\tilde{z}_1^\dagger \tilde{z}_1 - (F(\hat{N}_1 + k_1) - F(\hat{N}_1))\tilde{z}_2 \tilde{z}_2^\dagger . \tag{4.7}$$

From (4.5) and (4.7) we see that it is convenient to define $F(\theta) = \theta$, because we get

$$[\tilde{z}_i, \tilde{z}_i^\dagger] = k_1 \qquad [\tilde{K}_5, \tilde{K}_6] = -k_2 \hat{N}_1 - k_1 \hat{N}_2 - k_1 k_2 \qquad i = 1, 2. \tag{4.8}$$

However, we must be careful and observe that the relations (4.8) are not valid when applied to states $|n_1, n_2\rangle$ with $n_1 < k_1$ or $n_2 < k_2$. To have commutation relations of creation and annihilation operators which are valid in the complete Hilbert space generated by the states $|n_1, n_2\rangle$, one must make the definitions [7]

$$\bar{\tilde{z}}_i^\dagger = \left(\left\lfloor \frac{\hat{N}_i}{k_i} \right\rfloor \frac{(\hat{N}_i - k_i)!}{(N_i)!} \right)^{1/2} \left(\hat{z}_i^\dagger\right)^{k_i} \tag{4.9a}$$

$$\bar{\tilde{z}}_i = (\hat{z}_i)^{k_i} \left(\left\lfloor \frac{\hat{N}_i}{k_i} \right\rfloor \frac{(\hat{N}_i - k_i)!}{(N_i)!} \right)^{1/2} \tag{4.9b}$$

where $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$. From (4.9) it is easy to check that

$$\bar{\tilde{N}}_i = \bar{\tilde{z}}_i^\dagger \bar{\tilde{z}}_i = \left\lfloor \frac{\hat{N}_i}{k_i} \right\rfloor . \tag{4.10}$$

Then the Lie algebra is identified by considering the following operators

$$\hat{h}_1 = \bar{\tilde{N}}_1 - \bar{\tilde{N}}_2 \tag{4.11a}$$

$$\bar{\tilde{K}}_5 = \bar{\tilde{z}}_1^\dagger \bar{\tilde{z}}_2^\dagger \tag{4.11b}$$

$$\bar{\tilde{K}}_6 = \bar{\tilde{z}}_1 \bar{\tilde{z}}_2 \tag{4.11c}$$

$$C_1 = \tfrac{1}{2}(\bar{\tilde{N}}_1 + \bar{\tilde{N}}_2 + 1) \tag{4.11d}$$

that satisfy the commutation relations

$$[C_1, \bar{\tilde{K}}_5] = \bar{\tilde{K}}_5 \qquad [C_1, \bar{\tilde{K}}_6] = -\bar{\tilde{K}}_6 \qquad [\bar{\tilde{K}}_5, \bar{\tilde{K}}_6] = -2C_1 . \tag{4.12}$$

These were evaluated by using the fact that $[\bar{\tilde{z}}_i, \bar{\tilde{z}}_j^\dagger] = \delta_{ij}$, for any state $|n_1, n_2\rangle$, and they are the generators of a u(1, 1) Lie algebra, with $\hat{h}_1$ generating the invariant sub-algebra.

(iii) Finally for $-1 < \lambda < 1$, the symmetry algebra is constituted by the quantum analogues of the set of constants of motion (3.19). They satisfy the following commutation relations:

$$[\hat{m}_2, \hat{K}_3] = \hat{K}_3 \qquad [\hat{m}_2, \hat{K}_4] = -\hat{K}_4 \tag{4.13a}$$

$$[\hat{K}_3, \hat{K}_4] = \frac{\hat{N}_1!}{(\hat{N}_1 - k_1)!}\frac{(\hat{N}_2 + k_2)!}{\hat{N}_2!} - \frac{(\hat{N}_1 + k_1)!}{\hat{N}_1!}\frac{\hat{N}_2!}{(\hat{N}_2 - k_2)!} \tag{4.13b}$$

and all the others commutators are zero. They generate an algebra, but to give a Lie algebra a redefinition of the operators $\hat{K}_3$ and $\hat{K}_4$ must be done. Proceeding as in the previous case, i.e. using the creation and annihilation operators defined in (4.9), we consider the operators

$$\bar{\hat{K}}_3 = \bar{\hat{z}}_1^\dagger \bar{\hat{z}}_2 \qquad \bar{\hat{K}}_4 = \bar{\hat{z}}_2^\dagger \bar{\hat{z}}_1. \tag{4.14}$$

Evaluating the commutation relations between the operators

$$\hat{h}_2 = \bar{\hat{N}}_1 + \bar{\hat{N}}_2 \qquad C_2 = \tfrac{1}{2}(\bar{\hat{N}}_1 - \bar{\hat{N}}_2) \tag{4.15}$$

together with $\bar{\hat{K}}_3$ and $\bar{\hat{K}}_4$ we have

$$[C_2, \bar{\hat{K}}_3] = \bar{\hat{K}}_3 \qquad [C_2, \bar{\hat{K}}_4] = -\bar{\hat{K}}_4 \qquad [\bar{\hat{K}}_3, \bar{\hat{K}}_4] = 2C_2 \tag{4.16}$$

and the operator $\hat{h}_2$ is the ideal of the algebra. Thus we get for this case a u(2) symmetry Lie algebra.

5. Conclusions

We have established a procedure that uses Noether's theorem to find the symmetry Lie algebra of a quantum system with accidental degeneracy. The main foundations are the following. Firstly, to solve the differential equations that determine the constants of motion (cf equation (3.11)). Secondly, once we have chosen the minimal set of constants of motion that are closed under Poisson brackets, to identify the classical Lie algebra one needs in general to form combinations of the selected Noether charges. Thirdly, to find the corresponding quantum counterparts. Afterwards, the identification of the quantum symmetry Lie algebra can be done immediately by making the standard replacement of Poisson brackets by commutators. However, this is only true if there are not ambiguities in establishing the associated quantum operators for the constants of motion which form a Lie algebra under the Poisson bracket operation. If this is not the case, it is more convenient to choose the minimal set of constants of motion that allows a quantum extension, and make the necessary redefinitions to build the associated Lie algebra of the system. Following this procedure we get for the generalized anisotropic two-dimensional harmonic oscillator (1.5) the symmetry algebras which determine the degeneracy of the system. The symmetry Lie algebras are, depending on the value for λ, $\mathrm{w}(1) \oplus \mathrm{u}(1)$, u(2), and u(1, 1). However with the generators of the first one a Holstein–Primakoff realization [11] of an u(1, 1) Lie algebra can be obtained.

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Appendix A

The Poisson brackets between N_1 and N_2 are zero, and

$$\begin{aligned} &\{N_1, K_3\} = -ik_1 K_3 \qquad \{N_2, K_3\} = ik_2 K_3 \\ &\{N_1, K_4\} = ik_1 K_4 \qquad \{N_2, K_4\} = -ik_2 K_4 \\ &\{K_3, K_4\} = -i\left(k_2^2 N_1 - k_1^2 N_2\right) {N_1}^{k_1-1} {N_2}^{k_2-1}. \end{aligned} \tag{A1}$$

Let us choose normalization functions $F_i = F_i(N_1, N_2)$ such that by defining

$$\tilde{K}_i = F_i K_i \qquad i = 3, 4 \tag{A2}$$

we obtain

$$\{\tilde{K}_3, \tilde{K}_4\} = i\frac{C}{k_1 + k_2}(N_1 - N_2) \tag{A3}$$

where C is a constant, and the quantity on the right-hand side, the angular momentum, is always independent of the Hamiltonian. A direct calculation gives

$$\begin{aligned} \{F_3 K_3, F_4 K_4\} = &-iF_3 F_4\left(k_2^2 N_1 - k_1^2 N_2\right) N_1^{k_1-1} N_2^{k_2-1} \\ &+ i\left(k_1 \frac{\partial F_3 F_4}{\partial N_1} - k_2 \frac{\partial F_3 F_4}{\partial N_2}\right) N_1^{k_1} N_2^{k_2}. \end{aligned} \tag{A4}$$

Substituting the last expression into equation (A3), we get a differential equation for $F_3 F_4$, and we propose, for it, a solution of the form

$$F_3 F_4 = N_1^{-k_1} N_2^{-k_2} P(N_1, N_2). \tag{A5}$$

In this way we get

$$-k_2 \frac{\partial P}{\partial N_2} + k_1 \frac{\partial P}{\partial N_1} = \frac{C}{k_1 + k_2}(N_1 - N_2). \tag{A6}$$

Making the change of variables

$$h_2 = \frac{1}{k_1} N_1 + \frac{1}{k_2} N_2 \qquad M = N_1 - N_2 \tag{A7}$$

the equation (A6) is rewritten as

$$(k_1 + k_2)\frac{\partial P}{\partial M} = CM \tag{A8}$$

whose solution is immediately seen to be

$$P = \frac{C}{2(k_1 + k_2)} M^2 + g(h_2). \tag{A9}$$

Choosing $g(h_2) = 0$ we get the solution (3.23).

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